

AP Calculus BC
Chapter 9 Test 1 Review Outline

Sequences: $\{a_n\}$

1. Listing the terms of a sequence
2. Finding a formula for the nth term of a sequence
3. Determining the convergence/divergence of a sequence

Series: $\sum_{k=1}^{\infty} a_k$

1. Geometric: $\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \dots, a \neq 0, r \neq 0$

a. Converges if $|r| < 1$, diverges otherwise

b. If it converges, the sum = $\frac{a}{1-r}$

2. Harmonic: $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ (diverges)

3. P-series: $\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$

a. Converges if $p > 1$

b. Diverges if $0 < p \leq 1$

4. Alternating:

- a. Be able to approximate the sum of an alternating series
- b. Error of approximation is less than the next unused term

c. Absolute convergence: $\sum_{k=1}^{\infty} a_k$ converges absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges.

d. Conditional convergence: If $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ does not

e. If a series converges absolutely, then it converges (two for one.)

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Tests for Convergence of Series

1. Geometric series – see above
2. p-series test (including harmonic) – see above
3. Divergence/nth term test: If $\lim_{k \rightarrow \infty} a_k \neq 0 \Rightarrow \sum_{k=1}^{\infty} a_k$ diverges.
4. Integral test: $\int_c^{\infty} f(x)dx$ and $\sum_{k=1}^{\infty} a_k$ both converge or both diverge
5. Ratio test: If $\rho = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} < 1$ then the series converges, if $\rho > 1$ then it diverges, if $\rho = 1$ then the test is inconclusive.
6. Root test: If $\rho = \lim_{k \rightarrow \infty} (a_k)^{\frac{1}{k}} < 1$ then the series converges, if $\rho > 1$ then it diverges, if $\rho = 1$ then the test is inconclusive.
7. Limit comparison test: If $\rho = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$ is finite and greater than 0, then both series converge or both diverge.
8. Comparison test: If $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k$ term by term, then
 - a. If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges. (If the bigger converges, the smaller must.)
 - b. If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ diverges. (If the smaller diverges, the bigger must.)
9. Alternating series test: an alternating series converges if
 - a. $a_1 > a_2 > a_3 \dots$, i.e. the sequence $\{a_n\}$ is decreasing (prove), and
 - b. $\lim_{k \rightarrow \infty} a_k = 0$
10. Ratio test for absolute convergence: If $\rho = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$ then the series converges absolutely, if $\rho > 1$ then it diverges, if $\rho = 1$ then the test is inconclusive.

Summary of the convergence tests that may appear on the AP Calculus BC exam.

Test Name	The series ...	will converge if	Or will diverge if	Comments
n^{th} –term test	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	For divergence only; the converse is false.
Geometric	$\sum_{n=1}^{\infty} ar^{n-1}$	$-1 < r < 1$	$r \leq -1 \text{ or } r \geq 1$	Sum = $\frac{a}{1-r}$
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$ a_{n+1} < a_n $ and $\lim_{n \rightarrow \infty} a_n = 0$		Error bound $ S_{\infty} - S_n < a_{n+1} $
Integral test	$\sum_{n=1}^{\infty} a_n$ and $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x) dx$ converges	$\int_1^{\infty} f(x) dx$ diverges	f must be continuous, positive and decreasing.
p -series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Direct comparison	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{ a_{n+1} }{ a_n } < 1$	$\lim_{n \rightarrow \infty} \frac{ a_{n+1} }{ a_n } > 1$	If $\lim_{n \rightarrow \infty} \frac{ a_{n+1} }{ a_n } = 1$ the ratio test cannot be used.

Other useful convergence tests that may be used.

Test Name	The series ...	will converge if	Or will diverge if	Comments
Limit Comparison	$\sum_{n=1}^{\infty} a_n$	$a_n > 0, b_n > 0$ $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$a_n > 0, b_n > 0$ $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} < 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} > 1$	The test cannot be used if $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$

In Exercises 51–68, determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.

$$51. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5}{n}$$

$$52. \sum_{n=1}^{\infty} \frac{5}{n}$$

$$53. \sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$$

$$54. \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$$

$$55. \sum_{n=1}^{\infty} \frac{2n}{n+1}$$

$$56. \sum_{n=1}^{\infty} \frac{n}{2n^2 + 1}$$

$$57. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n}$$

$$58. \sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$$

$$59. \sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$$

$$60. \sum_{n=1}^{\infty} \frac{2^n}{4n^2 - 1}$$

$$61. \sum_{n=1}^{\infty} \frac{\cos n}{2^n}$$

$$62. \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

$$63. \sum_{n=1}^{\infty} \frac{n^7 n}{n!}$$

$$64. \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

$$65. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$$

$$66. \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$$

Summary of Tests for Series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$ r < 1$	$ r \geq 1$	Sum: $S = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n \rightarrow \infty} b_n = L$		Sum: $S = b_1 - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n \rightarrow \infty} a_n = 0$		Remainder: $ R_N \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x) dx$ converges	$\int_1^{\infty} f(x) dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x) dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } < 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } > 1$	Test is inconclusive if $\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right < 1$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right > 1$	Test is inconclusive if $\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

$$(51) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 5}{n} \rightarrow \text{CONVERGES CONDITIONALLY}$$

ALTERNATING SERIES TEST -

MAGNITUDE OF TERMS DECREASE

$$\text{TO } 0 \Rightarrow \sum \frac{(-1)^{n+1} \cdot 5}{n} \text{ CONV.}$$

$$\text{BUT } \sum_{n=1}^{\infty} |(-1)^{n+1} \cdot \frac{5}{n}| = \sum_{n=1}^{\infty} \frac{5}{n}$$

$$= \sum_{n=1}^{\infty} \frac{5}{n}$$

$$= 5 \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \text{HARMONIC} \Rightarrow \text{DIVERGES}$$

$$(52) \sum_{n=1}^{\infty} \frac{5}{n} \text{ DIVERGES (SEE #51)}$$

$$(53) \sum_{n=1}^{\infty} \frac{3}{n \cdot n^{1/2}} \text{ CONVERGES}$$

$$P\text{-SERIES TEST: } p = 3/2 \Rightarrow \text{CONV.}$$

$$3 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$(54) \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n \text{ GEOMETRIC, } r = \frac{\pi}{4} \Rightarrow \text{CONV.}$$

$$(55) \sum_{n=1}^{\infty} \frac{2^n}{n+1} \text{ DIVERGES}$$

DIVERGENCE TEST:

$$\lim_{n \rightarrow \infty} \frac{2^n}{n+1} = 2 \neq 0$$

$$(56) \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} \text{ DIVERGES.}$$

LCT: COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n}$ HARMONIC DIVERGES

$$L = \lim_{n \rightarrow \infty} \frac{n}{2^{n+1}} \cdot \frac{n}{1} = \frac{1}{2}$$

$$(57) \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} \text{ DIVERGES}$$

DIVERGENCE TEST

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n 3^{n-2}}{2^n} \right| = \infty \neq 0$$

$$(58) \frac{10}{3} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ CONVERGES P-SERIES, } p = 3/2 > 1$$

$$(59) \sum_{n=1}^{\infty} \frac{10n+3}{n \cdot 2^n} \text{ CONVERGES}$$

RATIO TEST:

$$p = \lim_{n \rightarrow \infty} \left| \frac{10(n+1)+3}{(n+1)2^{n+1}} \cdot \frac{n \cdot 2^n}{10n+3} \right| = \frac{1}{2} < 1$$

$$(60) \sum_{n=1}^{\infty} \frac{2^n}{4n^2-1} \text{ DIVERGES}$$

RATIO TEST:

$$p = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{4(n+1)^2-1} \cdot \frac{4n^2-1}{2^n} \right| = 2 > 1$$

(61) $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ CONVERGES

(63) $\sum_{n=1}^{\infty} \frac{n \cdot 7^n}{n!}$ DIVERGES

DIRECT COMPARISON TEST:

COMPARE TO $\sum \frac{1}{2^n}$ GEOMETRIC, $r = \frac{1}{2}$
 \Rightarrow CONVERGES

$$\cos n \leq 1$$

$$\frac{\cos n}{2^n} \leq \frac{1}{2^n}$$

$\Rightarrow \sum \frac{\cos n}{2^n}$ CONVERGES

(62) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ CONVERGES
 CONDITIONALLY

INTEGRAL TEST (ABS. CONVERGENCE)

$$\int_2^{\infty} \frac{1}{x \ln x} dx \quad u = \ln x \quad du = \frac{1}{x} dx$$

$$\int_{\ln 2}^{\infty} \frac{1}{u} du = [\ln u] \Big|_{\ln 2}^{\infty} = \infty$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ DIVERGES}$$

BUT...

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n} \text{ CONVERGES BY THE ABS. SERIES TEST -}$$

THE MAGNITUDES OF TERMS

ARE DECREASING TO 0.

RATIO TEST:

$$p = \lim_{n \rightarrow \infty} \frac{(n+1)^{-1}}{(n)^{-1}} \cdot \frac{x^n}{n!} = 7 > 1$$

(64) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ CONVERGES

INTEGRAL TEST (PARTS)

$$\int_1^{\infty} x^{-2} \ln x dx \quad u = \ln x \quad du = \frac{1}{x} dx \quad v = x^{-1}$$

$$= -\frac{1}{x} \ln x + \int x^{-2} dx$$

$$= -\frac{1}{x} \ln x - \frac{1}{x}$$

$$= -\frac{\ln x + 1}{x} \Big|_1^{\infty} = 0 + \frac{\ln 1 + 1}{1} = 1$$

(65) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n!}$ CONVERGES
 ABSOLUTELY

RATIO TEST (ABSOLUTE CONVERGENCE)

$$p = \lim_{n \rightarrow \infty} \left| \frac{3^n}{(n+1)!} \cdot \frac{n!}{3^n} \right| = 0 < 1$$

(66) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3^n}{n 2^n}$ DIVERGES

RATIO TEST (ABS. CONVERGENCE)

$$p = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n \cdot 2^n}{3^n} \right| = \frac{3}{2} > 1$$